RAPID COMMUNICATIONS

PHYSICAL REVIEW E **90**, 020901(R) (2014)

Class of compound dissipative solitons as a result of collisions in one and two spatial dimensions

Orazio Descalzi[1,2,*] and Helmut R. Brand[2]

[1]*Complex Systems Group, Facultad de Ingeniería y Ciencias Aplicadas, Universidad de los Andes, Avenida Monseñor Álvaro del Portillo 12.455, Las Condes, Santiago, Chile*

[2]*Department of Physics, University of Bayreuth, 95440 Bayreuth, Germany*

(Received 14 May 2014; published 4 August 2014)

We study the interaction of quasi-one-dimensional (quasi-1D) dissipative solitons (DSs). Starting with quasi-1D solutions of the cubic-quintic complex Ginzburg-Landau (CGL) equation in their temporally asymptotic state as the initial condition, we find, as a function of the approach velocity and the real part of the cubic interaction of the two counterpropagating envelopes: interpenetration, one compound state made of both envelopes or two compound states. For the latter class both envelopes show DSs superposed at two different locations. The stability of this class of compound states is traced back to the quasilinear growth rate associated with the coupled system. We show that this mechanism also works for 1D coupled cubic-quintic CGL equations. For quasi-1D states that are not in their asymptotic state before the collision, a breakup along the crest can be observed, leading to nonunique results after the collision of quasi-1D states.

DOI: 10.1103/PhysRevE.90.020901 PACS number(s): 82.40.Bj, 42.65.Sf, 05.70.Ln, 47.20.Ky

The study presented here is part of the field of stable localized solutions in driven and damped nonequilibrium systems. Frequently these localized solutions are called dissipative solitons (DSs) [1] to emphasize that dissipation and driving are essential for their stable existence. They have been analyzed theoretically [2–31] and experimentally [32–43] for a variety of systems. It emerges, however, that the number of studies dedicated to the interaction of counterpropagating DSs is much smaller [3–5,9–12,15,21–23,26,30,34,35,38,43].

One class of dissipative solitons is quasi-one-dimensional (quasi-1D) solutions, which are localized in one spatial dimension, but spatially extended in the other. Their stable existence over a range of parameters has been found first for the envelope equation applicable to a laser with a saturable absorber in the field of nonlinear optics [6]. Subsequently this type of behavior has been shown to arise as well for other physical systems [12,18,19,24,30]. Recently we have analyzed quasi-1D solutions for the cubic-quintic complex Ginzburg-Landau (CGL) equation [30]. We have shown that, in addition to stationary quasi-1D solutions, there are oscillatory quasi-1D DSs that are reached via a forward Hopf bifurcation. Furthermore, we demonstrated that there are stable compound states made up of a combination of a quasi-1D solution and a DS localized in two dimensions (2D), namely, with a radially symmetric 2D stationary DS and with an azimuthally symmetric exploding DS. We have pointed out [30] that the experiments performed by Miranda and Burguete [44,45] on an array of convective oscillators show several common features with the results obtained from a cubic-quintic CGL equation.

Stimulated by recent experiments in a bioinspired system on the interaction of localized wall-like objects [46], we investigate here the interaction of counterpropagating quasi-1D DSs for coupled cubic-quintic CGL equations for envelopes A and B. In addition to interpenetration of the type also observed in Ref. [46] with the size and shape of the quasi-1D states unchanged after interpenetration, we find two types of compound states. One of these compound states consists of two quasi-1D localized objects containing $|A|$ and $|B|$ simultaneously. We show that this class of compound dissipative solitons also stably exists for two coupled cubic-quintic CGL equations in 1D and we present a simple argument for their stable existence.

We investigate two coupled complex subcritical cubic-quintic Ginzburg-Landau equations for counterpropagating waves,

$$\partial_t A - v\partial_x A = \mu A + (\beta_r + i\beta_i)|A|^2 A + (\gamma_r + i\gamma_i)|A|^4 A + (c_r + ic_i)|B|^2 A + (D_r + iD_i)\Delta_2 A, \quad (1)$$

$$\partial_t B + v\partial_x B = \mu B + (\beta_r + i\beta_i)|B|^2 B + (\gamma_r + i\gamma_i)|B|^4 B + (c_r + ic_i)|A|^2 B + (D_r + iD_i)\Delta_2 B, \quad (2)$$

where $A(x,y,t)$ and $B(x,y,t)$ are complex fields and where we have discarded quintic cross-coupling terms for simplicity. A and B are slowly varying envelopes. $\Delta_2 = (\partial_x^2 + \partial_y^2)$ is the 2D Laplacian. The fast spatial and temporal variations have already been split off when writing down the coupled envelope equations. To compare with measurable quantities such as, for example, temperature variations in fluid dynamics, these rapid variations must be taken into account [47–49].

We have carried out our numerical studies for the following values of the parameters, which we kept fixed for the present purposes: $\mu = -0.9$, $\beta_r = 1$, $\beta_i = 0.8$, $\gamma_r = -0.1$, $\gamma_i = -0.6$, $D_r = 0.125$, $D_i = 0.5$, and $c_i = 0$. Our parameters satisfy $D_i > 0$, which means we are in the anomalous linear dispersion regime. But compound states also exist for $D_i < 0$. β_r must be positive and γ_r negative in order to guarantee the existence of two homogeneous attractors (subcritical bifurcation). Stable pulses exist when the cubic-quintic Ginzburg-Landau equation becomes nonvariational. Thus, one of the parameters (β_i,γ_i,D_i) must be different from zero. We have plotted in Fig. 1 the phase diagram using the approach velocity v of the quasi-1D states in the x direction and the strength of the cubic cross coupling of counterpropagating waves c_r

---

[*] Corresponding author: odescalzi@miuandes.cl

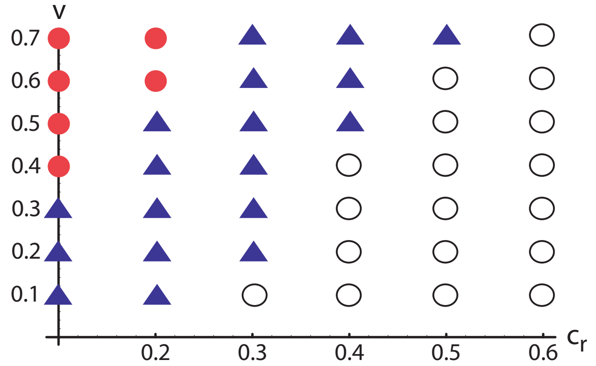

FIG. 1. (Color online) Phase diagram in the plane approach velocity v vs strength of cubic cross coupling of counterpropagating waves c_r for destabilizing (positive) values of c_r. Red solid circles (●) mark the interpenetration of two quasi-1D states. Blue solid triangles (▲) depict a compound state of two quasi-1D states as the result of the collision, and for the region marked with open circles (○) we obtain two compound states, both involving both envelopes.

as the axes. Here we focus on destabilizing (positive) values of c_r.

For the simulation of Eqs. (1) and (2), as a numerical method, we use explicit fourth-order Runge-Kutta finite differencing with a rectangular grid of 700 points in x and 350 points in y along a grid spacing of $dx = dy = 0.1$ (corresponding to a rectangular box size 70 × 35) and a time step $dt = 0.005$. The stability of the solutions has been tested by introducing a small amount of noise and waiting long enough to avoid transients. The initial conditions were carefully prepared so that the counterpropagating waves A and B were already in the asymptotic regime [see Fig. 2(a)].

In Fig. 1 we present an overview of the types of behavior as a function of the approach velocity v and the destabilizing cross coupling c_r. The range covered in the plot is $0 < v < 0.7$ and $0 < c_r < 0.6$. Without noise three characteristic types of behavior emerge as the results of collisions of two quasi-1D states. For v sufficiently large and c_r sufficiently small, interpenetration is obtained with the final quasi-1D states unchanged in size and shape from the initial state. For v sufficiently small and c_r sufficiently large, we obtain two compound states (see the discussion below and Fig. 2). In the range between these two limiting cases the result of collisions of two quasi-1D states is one compound state (see the discussion below and Fig. 3).

In Fig. 2 we have plotted the initial and the final result starting with two approaching quasi-1D states leading, as a result of the collision, to two compound states: as soon as the interaction starts $|A|$ is growing at the location of $|B|$ and vice versa. The dynamics will be further elucidated below when we discuss the one-dimensional case in Fig. 4. Its mechanism is rather straightforward. When inspecting the quasilinear growth rate

$$\dot{A} = \sigma A \quad \text{with} \quad \sigma = (\mu + c_r|B|^2), \tag{3}$$

one sees that for destabilizing c_r sufficiently large a positive growth rate can effectively result for the region where $|B|$ acquires nonvanishing values and vice versa for $|A|$ and $|B|$ interchanged. Provided that the other parameters in the equation do not lead to a filling in of the whole system (this case has been discussed before in 1D [4] and for circularly symmetric stationary DSs in 2D [5]) we obtain as final states two quasi-1D states containing $|A|$ as well as $|B|$ with equal amplitudes.

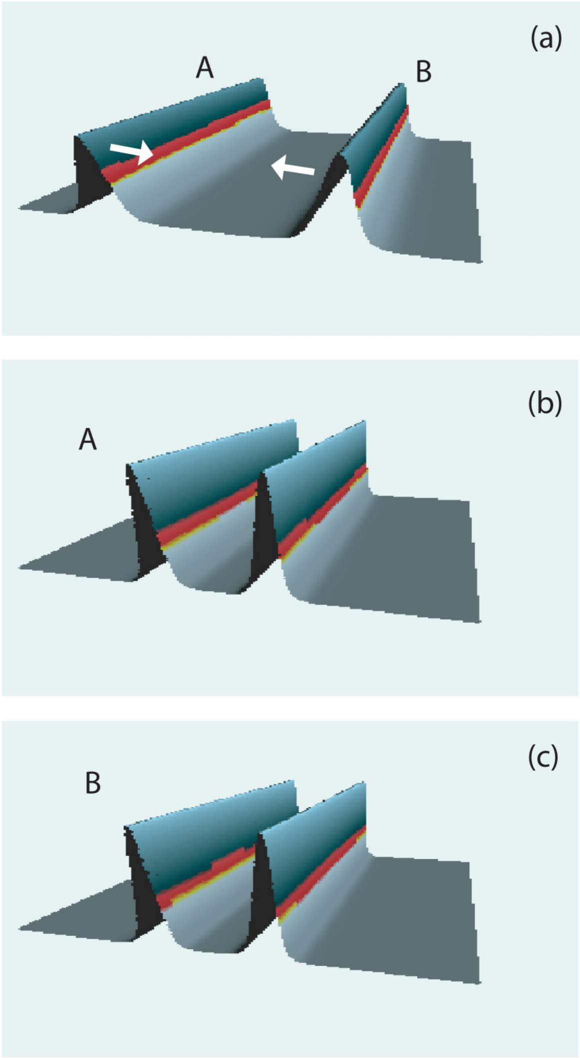

FIG. 2. (Color online) (a) shows the two quasi-1D states in the asymptotic regime before they interact. (b) and (c) show the final bound state consisting of two compound states of $|A|$ and $|B|$. For clarity, we have plotted $|A|$ and $|B|$ separately, underlining that both amplitudes assume the same shape at both locations of the DSs. This bound object we denote as two compound states. The results show are for $v = 0.4$ and $c_r = 0.5$.

In the intermediate parameter range for approach velocity and destabilizing cross coupling—for example, for a sufficiently small approach velocity—we obtain the situation depicted for the final result in Fig. 3. This situation is the analog of the corresponding situation in 1D [4] and for circularly symmetric states in 2D [5]. We note that for this one compound state situation the peaks for $|A|$ and $|B|$ are typically not at the same location, but slightly shifted with respect to each other.

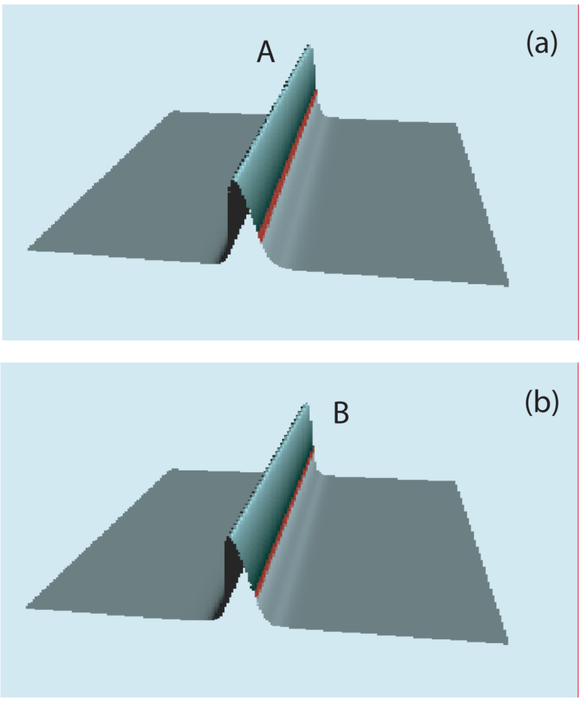

FIG. 3. (Color online) Starting with two quasi-1D states as the initial conditions of the same type as in Fig. 2(a), we show the final state obtained for $v = 0.5$ and $c_r = 0.4$. In (a) we have plotted $|A|$ and in (b) we have plotted $|B|$, demonstrating a compound state with $|A|$ and $|B|$ having the same shape at almost the same location: one compound state.

To analyze the time evolution to reach two compound states in more detail we have complemented our investigations in 2D on the interaction of quasi-1D states of two coupled cubic-quintic CGL equations by studying the interaction of fixed shape DSs in one spatial dimension. It turns out that the corresponding phase diagram for v and c_r nearly superposes on that given in Fig. 1 for quasi-1D collisions. In Fig. 4 we show x-t plots and three snapshots for the time evolution of the interaction for the same parameter values as for Fig. 2, $v = 0.4$ and $c_r = 0.5$. From the inspection of the x-t plots for $|A|$ and $|B|$ shown in Figs. 4(a) and 4(b), as well as from the three snapshots given in Figs. 4(c)–4(e), we can infer immediately that the growth of $|A|$ at the location of the DS in $|B|$ (and vice versa) takes place during a fairly short time interval. The fact that the time evolution for 1D DSs and for quasi-1D DSs is parallel also supports the simple mechanism suggested above via the quasilinear growth rate outlined above as the key ingredient for the generation of this class of stable compound DSs.

In this connection we point out that in the cases studied so far mainly two time scales come into play for the interaction of the quasi-1D and the 1D DSs: the time scale associated with the approach velocity v and the time scale set by the pattern growth via the quasilinear growth rate.

To find out how important it is to start the investigation of the collision process with the asymptotic shape in the time of the quasi-1D solutions, we have also done some runs for which the asymptotic regime had not been reached

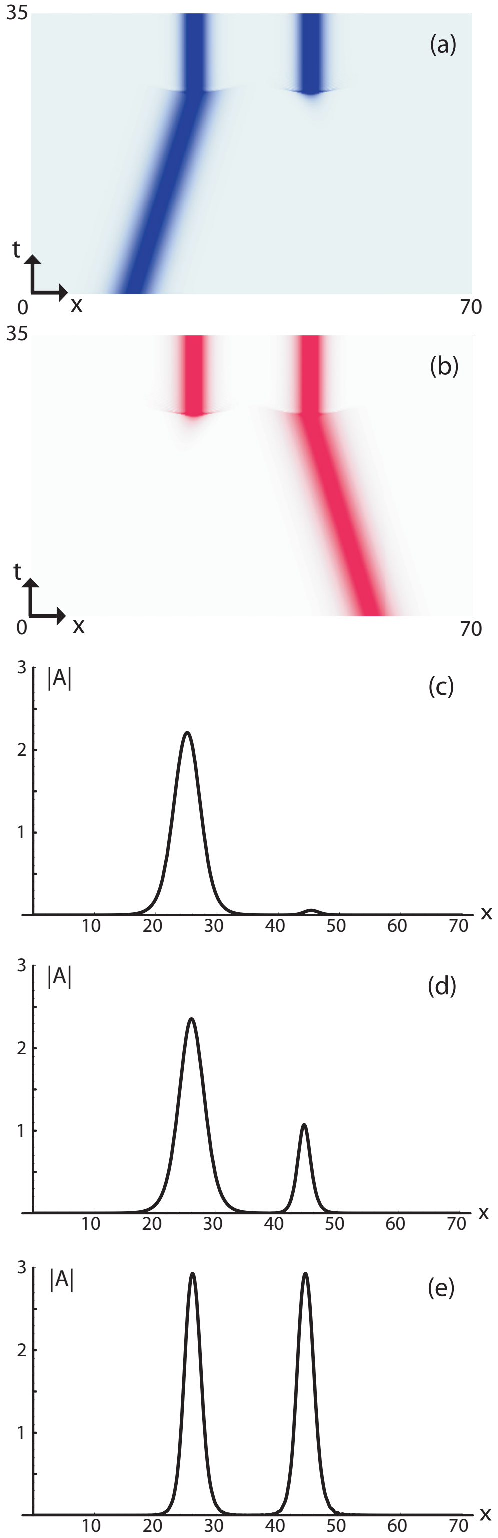

FIG. 4. (Color online) A two compound state resulting from two coupled 1D cubic-quintic CGL equations for $v = 0.4$ and $c_r = 0.5$. (a) shows the time evolution of $|A|$ as an x-t plot while (b) shows $|B|$ as a function of time. The box size is $L = 70$ and the duration of the x-t plot is $T = 35$. In (c)–(e) we show three snapshots demonstrating the growth of $|A|$ at the location of the $|B|$ DS as soon as the two stationary DSs interact. (c) $T = 22.5$, (d) $T = 24.7$, and (e) $T = 28.1$.

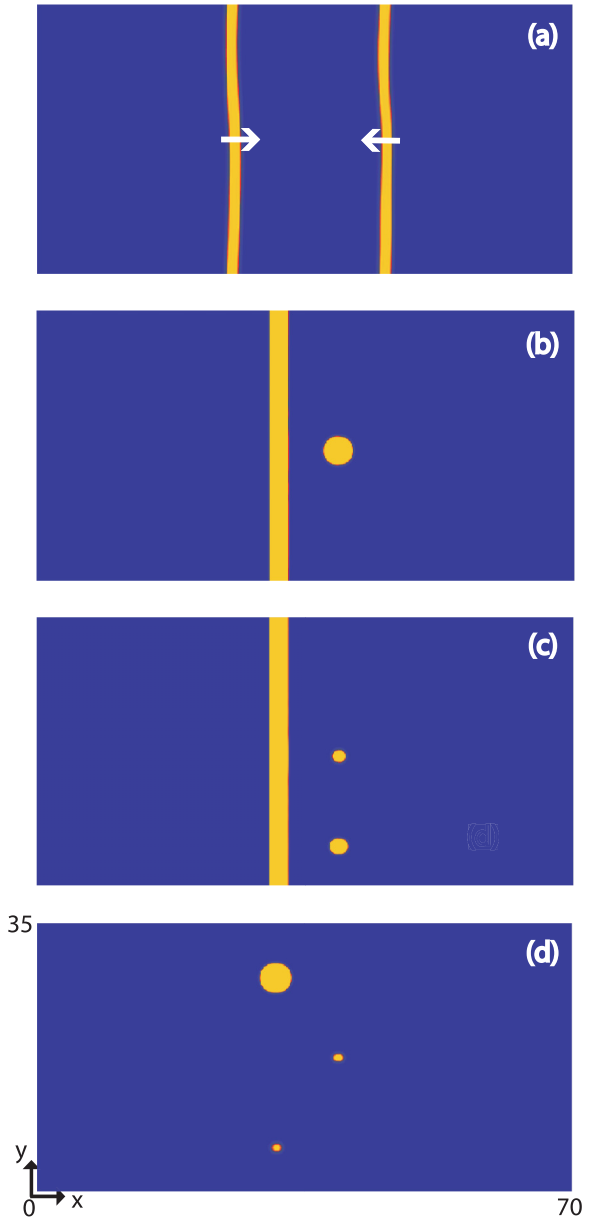

FIG. 5. (Color online) Snapshots of max(A,B) showing that the result of the collision of two quasi-1D states is not unique when the quasi-1D states initially have not reached their asymptotic shape. (a) Initial condition not in the asymptotic regime. (b)–(d) show three possible outcomes for the same parameters, $c_r = 0.5$ and $v = 0.4$: (b) a compound object of a quasi-1D state and one oscillating azimuthally symmetric 2D DS, (c) a compound object of a quasi-1D state and two oscillating (not in phase) azimuthally symmetric 2D DSs, and (d) three oscillating (not in phase) azimuthally symmetric compound 2D DSs.

completely [see Fig. 5(a)]. Some of the results of such runs for sufficiently small values of v and sufficiently large values of c_r are shown in Figs. 5(b)–5(d). Depending on minute differences between the not fully relaxed initial conditions, one obtains various types of outcomes: nonunique behavior. This nonuniqueness can be traced back to an instability along the crest of the quasi-1D states. This instability is associated with a third time scale competing with the other two just mentioned. Correspondingly, as a result of collisions for the same parameter values, one can get a number of outcomes including a compound state of a quasi-1D state interacting with one [Fig. 5(b)] or two [Fig. 5(c)] oscillating (not in phase) azimuthally symmetric 2D solutions, as well as three oscillating (not in phase) azimuthally symmetric compound 2D solutions [Fig. 5(d)].

In conclusion, we have demonstrated that collisions of quasi-1D solutions can lead to different results depending on the approach velocity, the value of the cubic cross coupling between counterpropagating envelopes, and the types of initial conditions. In particular, we have found a class of stable dissipative solitons: two compound states made up of both envelopes with equal weight. The stability of this class of solutions has been traced back to the effective quasilinear growth rate between counterpropagating envelopes. We have shown that this type of behavior arises over a large parameter range not only for the collisions of quasi-1D solutions in 2D, but also for fixed shape DSs in 1D for an almost identical part of parameter space, thus underlining the close connection between these two types of solutions.

Our present study opens the door to several areas of investigation. First of all, it will be important to see which features found for coupled cubic-quintic CGL equations can be found for other prototype equations such as order parameter equations or for equations of a reaction-diffusion type. For order parameter equations nonlinear optics emerges as a natural field while for reaction-diffusion systems chemical reactions and pattern formation in biological systems come to mind.

An outstanding challenge is certainly to find an experimental system which shows the types of interactions described here. Candidates include surface reactions of the type studied in Refs. [35,38] for which one has seen solitonlike behavior for arc-shaped (wall-like) objects, which can, in some cases, pass through each other in a solitonlike fashion. Quite recently an interesting candidate has emerged in the field of biological systems [46]. Kuwayama and Ishida have observed arc-shaped solitonlike structures in nonchemotactic mutants of the slime mold *Dictyostelium discoideum*. These objects showed constant velocity and fixed shape. Both quantities were preserved after the collisions, for which these objects were able to pass through each other. Since the system investigated is dissipative and driven, it appears natural to investigate to what extent there are parallel features between the biological system and the model studied here. Very recently a more microscopic model on the cell level has been presented to analyze the results presented in Ref. [50]. Our suggestion is that a coarse-grained model of the type presented here might be useful in characterizing the biological system, in particular, it would be worthwhile to check experimentally whether one can also get compound states of the two types described here for the mutants studied [46].

O.D. wishes to acknowledge the support of FONDECYT (Project No. 1140139) and Universidad de los Andes through FAI initiatives. H.R.B. thanks the Deutsche Forschungsgemeinschaft for partial support of this work.

[1] *Dissipative Solitons: From Optics to Biology and Medicine*, edited by N. Akhmediev and A. Ankiewicz (Springer, Heidelberg, 2008).
[2] O. Thual and S. Fauve, J. Phys. France **49**, 1829 (1988).
[3] H. R. Brand and R. J. Deissler, Phys. Rev. Lett. **63**, 2801 (1989).
[4] R. J. Deissler and H. R. Brand, Phys. Lett. A **146**, 252 (1990).
[5] R. J. Deissler and H. R. Brand, Phys. Rev. A **44**, R3411(R) (1991).
[6] H. R. Brand and R. J. Deissler, Physica A **204**, 87 (1994).
[7] R. J. Deissler and H. R. Brand, Phys. Rev. Lett. **72**, 478 (1994).
[8] W. N. Reynolds, J. E. Pearson, and S. Ponce-Dawson, Phys. Rev. Lett. **72**, 2797 (1994).
[9] R. J. Deissler and H. R. Brand, Phys. Rev. E **51**, R852(R) (1995).
[10] J. Kosek and M. Marek, Phys. Rev. Lett. **74**, 2134 (1995).
[11] R. J. Deissler and H. R. Brand, Phys. Rev. Lett. **74**, 4847 (1995).
[12] H. Sakaguchi and H. R. Brand, Physica D **97**, 274 (1996).
[13] V. V. Afanasjev, N. Akhmediev, and J. M. Soto-Crespo, Phys. Rev. E **53**, 1931 (1996).
[14] T. Ohta, Y. Hayase, and R. Kobayashi, Phys. Rev. E **54**, 6074 (1996).
[15] H. Sakaguchi and H. R. Brand, Physica D **117**, 95 (1998).
[16] J. M. Soto-Crespo, N. Akhmediev, and A. Ankiewicz, Phys. Rev. Lett. **85**, 2937 (2000).
[17] N. Akhmediev, J. M. Soto-Crespo, and G. Town, Phys. Rev. E **63**, 056602 (2001).
[18] H. Sakaguchi and B. A. Malomed, Physica D **159**, 91 (2001).
[19] H. Sakaguchi and B. A. Malomed, Physica D **167**, 123 (2002).
[20] Y. Hayase, O. Descalzi, and H. R. Brand, Phys. Rev. E **69**, 065201 (2004).
[21] O. Descalzi and H. R. Brand, Phys. Rev. E **72**, 055202 (2005).
[22] O. Descalzi, J. Cisternas, and H. R. Brand, Phys. Rev. E **74**, 065201 (R) (2006).
[23] O. Descalzi, J. Cisternas, P. Gutiérrez, and H. R. Brand, Eur. Phys. J.: Spec. Top. **146**, 63 (2007).
[24] J. Burke and E. Knobloch, Chaos **17**, 037102 (2007).
[25] J. M. Soto-Crespo, N. Akhmediev, N. Devine, and C. Mejia-Cortes, Opt. Express **16**, 15388 (2008).
[26] O. Descalzi, J. Cisternas, D. Escaff, and H. R. Brand, Phys. Rev. Lett. **102**, 188302 (2009).
[27] O. Descalzi and H. R. Brand, Phys. Rev. E **82**, 026203 (2010).
[28] C. Cartes, O. Descalzi, and H. R. Brand, Phys. Rev. E **85**, 015205 (2012).
[29] C. Cartes, J. Cisternas, O. Descalzi, and H. R. Brand, Phys. Rev. Lett. **109**, 178303 (2012).
[30] O. Descalzi and H. R. Brand, Phys. Rev. E **87**, 022915 (2013).
[31] N. Akhmediev, J. M. Soto-Crespo, and H. R. Brand, Phys. Lett. A **377**, 968 (2013).
[32] P. Kolodner, D. Bensimon, and C. M. Surko, Phys. Rev. Lett. **60**, 1723 (1988).
[33] J. J. Niemela, G. Ahlers, and D. S. Cannell, Phys. Rev. Lett. **64**, 1365 (1990).
[34] P. Kolodner, Phys. Rev. A **44**, 6466 (1991).
[35] H. H. Rotermund, S. Jakubith, A. von Oertzen, and G. Ertl, Phys. Rev. Lett. **66**, 3083 (1991).
[36] P. B. Umbanhowar, F. Melo, and H. L. Swinney, Nature (London) **382**, 793 (1996).
[37] V. B. Taranenko, K. Staliunas, and C. O. Weiss, Phys. Rev. A **56**, 1582 (1997).
[38] A. von Oertzen, A. S. Mikhailov, H. H. Rotermund, and G. Ertl, J. Phys. Chem. B **102**, 4966 (1998).
[39] O. Lioubashevski, Y. Hamiel, A. Agnon, Z. Reches, and J. Fineberg, Phys. Rev. Lett. **83**, 3190 (1999).
[40] S. T. Cundiff, J. M. Soto-Crespo, and N. Akhmediev, Phys. Rev. Lett. **88**, 073903 (2002).
[41] E. A. Ultanir, G. I. Stegeman, D. Michaelis, C. H. Lange, and F. Lederer, Phys. Rev. Lett. **90**, 253903 (2003).
[42] F. S. Merkt, R. D. Deegan, D. I. Goldman, E. C. Rericha, and H. L. Swinney, Phys. Rev. Lett. **92**, 184501 (2004).
[43] H. Ebata and M. Sano, Phys. Rev. Lett. **107**, 088301 (2011).
[44] M. A. Miranda and J. Burguete, Phys. Rev. E **78**, 046305 (2008).
[45] M. A. Miranda and J. Burguete, Phys. Rev. E **79**, 046201 (2009).
[46] H. Kuwayama and S. Ishida, Sci. Rep. **3**, 2272 (2013).
[47] A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969).
[48] H. R. Brand, P. S. Lomdahl, and A. C. Newell, Physica D **23**, 345 (1986); Phys. Lett. A **118**, 67 (1986).
[49] J. Burguete, H. Chaté, F. Daviaud, and N. Mukolobwiez, Phys. Rev. Lett. **82**, 3252 (1999).
[50] T. Ohta and S. Yamanaka, Progr. Exp. Theor. Phys. **1**, 011J01 (2014).