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Noisy localized structures induced by large noise

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We investigate the influence of large noise on the formation of localized patterns in the framework of the cubic-quintic complex Ginzburg-Landau equation. The interaction of localization and noise can lead to filling in or noisy localized structures for fixed noise strength. To focus on the interaction between noise and localization we cover a region in parameter space, in particular, subcriticality, for which stationary stable deterministic pulses do not exist. Possible experimental tests of the work presented for autocatalytic chemical reactions and bioinspired systems are outlined.

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Noise is ubiquitous in nature, both close to thermal equilibrium as well as in strongly driven systems. In many cases noise is small in the sense that its amplitude is much smaller than the associated deterministic amplitude of a certain observable. This is typically the case for systems close to thermal equilibrium. Noise is also well known to play an important role not only in physics and chemistry, but also in biology (for a recent review we refer to Ref. [1]). In biology noise of varying magnitude occurs for many different length and time scales. Examples include the functional role of noise in gene expression [2–4] and chemotaxis on a microscopic scale [5]. Substantial amounts of noise influence biological processes as diverse as the start of the cell cycle [6], signaling cascades [7], information transmission [8], and noise modulating chemicals for drug synergies [9].

Quantitative studies of the interaction of noise and pattern formation in spatially extended nonequilibrium systems are by comparison already much less frequent. When it comes to the influence of noise, whose amplitude is a substantial fraction (say ∼10% or so) of the deterministic amplitude, most experimental studies have been carried out on surface reactions under the influence of combined additive and multiplicative noise [10–14] or purely multiplicative noise [15]. For additive noise of a substantial magnitude it has been shown [16,17] that reentrant spatiotemporal intermittency could be induced as a function of noise strength.

In parallel it has turned out that even small amounts of noise can have a qualitative influence on various aspects of spatially localized solutions [frequently called dissipative solitons (DSs) [18]] in dissipative and driven nonequilibrium systems. For example, it has been demonstrated that even weak noise can induce a partial annihilation of counterpropagating dissipative solitons [19], a phenomenon that had been observed near the onset of binary fluid convection [20,21] as well as for surface reactions [22,23]. In addition, it has been shown that for a reaction diffusion system driven by noise, localized objects of finite lifetime can occur [24].

When it comes to the influence of weak noise on one dissipative soliton, we have shown recently that weak noise (of less than ∼1% noise amplitude) can induce explosions for dissipative solitons and can lead to various routes to reach explosions [25]. In this connection we note that exploding dissipative solitons are one of the most fascinating types of spatially localized solutions. They were found for anomalous linear dispersion in the cubic-quintic complex Ginzburg-Landau equation [26]. Afterwards they were investigated in detail experimentally and theoretically [27–34]. Quite recently their existence has also been reported for reaction diffusion systems [35]. Compare also [36] for a recent review.

Here we study the question of localized solutions in the framework of the cubic-quintic Ginzburg-Landau equation under the influence of fairly large noise in the parameter regime, which is so strongly subcritical that deterministic localized solutions are no longer stable. For fixed degree of subcriticality and fixed noise strength we find, over an interval of noise strengths, either noise-induced localized states, which typically appear intermittently as a function of time or a filling in (a noisy spatially homogeneous finite amplitude pattern). In this regime the final result cannot be predicted starting from random initial conditions. This regime arises in addition to the two expected regimes: a noisy pattern-free state for lower noise strength and a noisy filled-in finite amplitude solution.

The cubic-quintic complex Ginzburg-Landau equation with additive noise we investigate here is of the form

$$\partial_t A = \mu A + (\beta_r + i\beta_i)|A|^2 A + (\gamma_r + i\gamma_i)|A|^4 A + (D_r + i D_i)\partial_{xx} A + \eta\,\xi, \tag{1}$$

where $A(x,t)$ is a complex field, β_r is positive, and γ_r is negative in order to guarantee that the bifurcation is subcritical, but saturates to quintic order. The stochastic force $\xi(x,t)$ denotes white noise with the properties $\langle\xi\rangle = 0$, $\langle\xi(x,t)\,\xi(x',t')\rangle = 0$ and $\langle\xi(x,t)\,\xi^*(x',t')\rangle = 2\delta(x-x')\delta(t-t')$, where ξ^* denotes the complex conjugate of ξ.

In our numerical simulations we keep all parameters fixed except for μ, the distance from linear onset, and η, the noise strength. The parameter values are $\beta_r = 1$, $\beta_i = 0.8$, $\gamma_r = -0.1$, $\gamma_i = -0.6$, $D_r = 0.125$, and $D_i = 0.5$ (positive) corresponding to an anomalous dispersion regime. Stable pulses can only exist when the cubic-quintic Ginzburg-Landau equation becomes nonvariational. Thus, at least one of the parameters (β_i, γ_i, D_i) must be different from zero [37–40].

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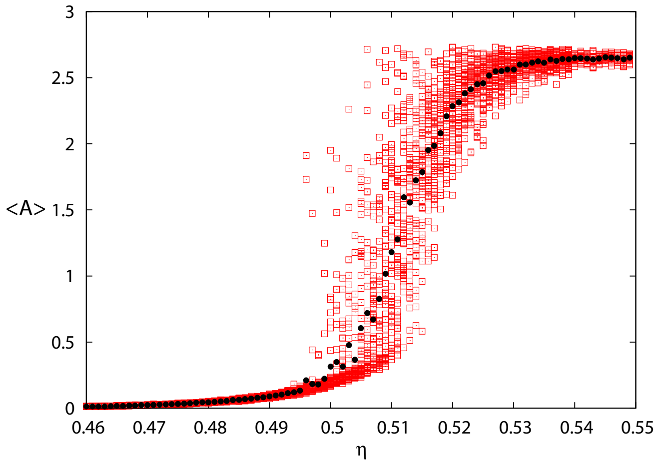

FIG. 1. (Color online) The amplitude averaged in time and space, $\langle A \rangle$ is plotted (black dots) as a function of the noise strength η for $\mu = -1.40$ for an average over 50 realizations, time $T = 10^4$ for each realization and a cut-off value 1.5 when taking the average over the amplitude.

The parameter μ is varied from -1.3 until -1.7. This range is chosen in a way to guarantee that there are no stable dissipative solitons for these strongly subcritical conditions, since the range of existence of stationary DSs ends at $\mu \sim -1.23$ [32].

In the discretized problem the stochastic force $\xi(x,t)$ is replaced by $(\chi_r + i\chi_i)/\sqrt{dx\,dt}$, where χ_r and χ_i are uncorrelated random numbers obeying a standard normal distribution.

Time integration of Eq. (1) was performed in all regions using a time-splitting pseudospectral scheme to compute the differential operator, with a box size $L = 50$ and $N = 625$, leading to a grid spacing of $dx = 0.08$ and a fourth order Runge-Kutta method to compute the nonlinear terms. The time step dt used was typically $dt = 0.005$. Throughout this Rapid Communication we used exclusively $A \equiv 0$, which is the lower deterministic attractor, as initial condition.

To characterize the noisy patterns quantitatively, we introduce $\langle A \rangle$, the average of $|A|$ in time and space,

$$\langle A \rangle = \frac{1}{LT} \int_0^T \int_0^L |A| dx\, dt. \quad (2)$$

To investigate the sensitivity of $\langle A \rangle$ plotted in Fig. 1 to changes in the parameters, we have varied the cut-off value (only values above a threshold value of $|A|$ are taken into account) of $|A|$ between 1.3 and 1.7 and the number of realizations between 20 and 100. It turns that the results shown are rather insensitive to these variations. Also varying the time scales of the run by one order of magnitude between $T = 5 \times 10^3$ and $T = 5 \times 10^4$ does not lead to any significant changes. $\langle A \rangle$, averaged over a large number of realizations, can be used as an order parameter, which varies smoothly as a function of the noise strength.

In Fig. 2 we have plotted snapshots of $R(x)$, the modulus of A, for fixed $\mu = -1.40$ for several values of the noise strength η. For small enough η we obtain a noisy zero solution without any spatial localization. Inspecting Figs. 2(a) and 2(b) for $\eta = 0.497$ we see that either noisy localized solution or a noisy zero solution arise. Increasing the noise strength to $\eta = 0.53$ we find that for fixed noise strength either a noisy localized solution [Fig. 2(c)] or a noisy filled in solution [Fig. 2(d)] emerge. For large enough noise strength only the noisy filled-in solution is obtained.

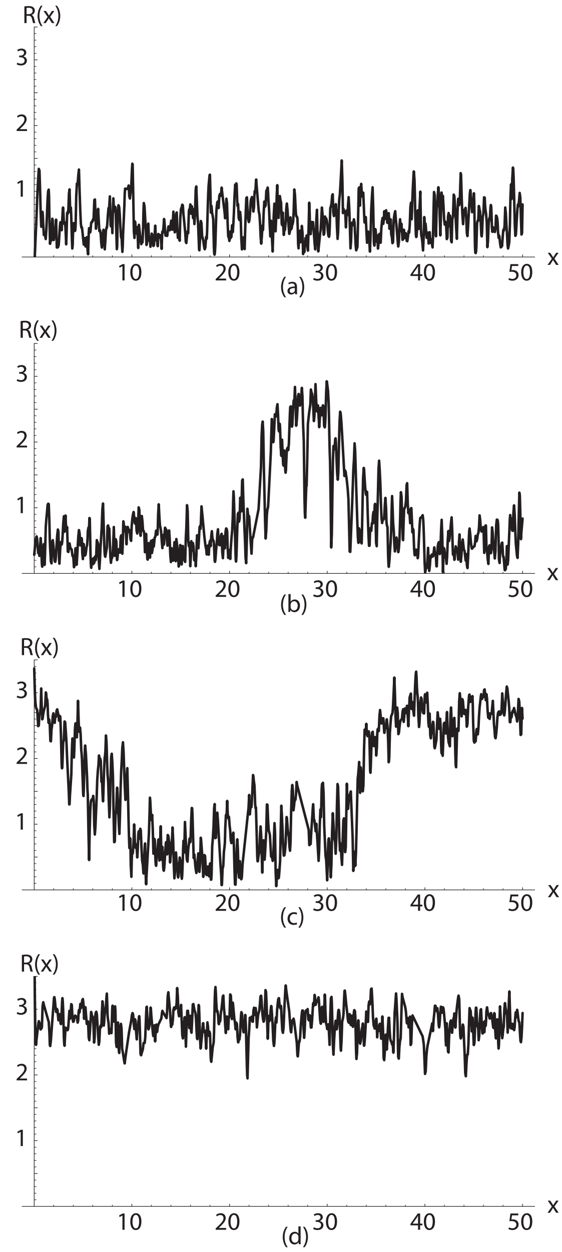

FIG. 2. The modulus of A, $R(x)$, is plotted as a function of space for $\mu = -1.40$ in the asymptotic time regime: (a) snapshot of a noisy zero solution for $\eta = 0.497$; (b) snapshot of a noisy localized state for $\eta = 0.497$; (c) snapshot of a noisy localized solution demonstrating very succinctly the coexistence between the upper and lower noisy attractors for $\eta = 0.53$; and (d) snapshot of a noisy filled-in finite amplitude solution for $\eta = 0.53$.

These results are further corroborated by the x-t plots shown in Fig. 3 for $T = 600$ in the asymptotic time regime, that is after initial transients have disappeared. As η increases from $\eta = 0.497$ [Fig. 3(a)] to $\eta = 0.53$ [Fig. 3(c)], we read off

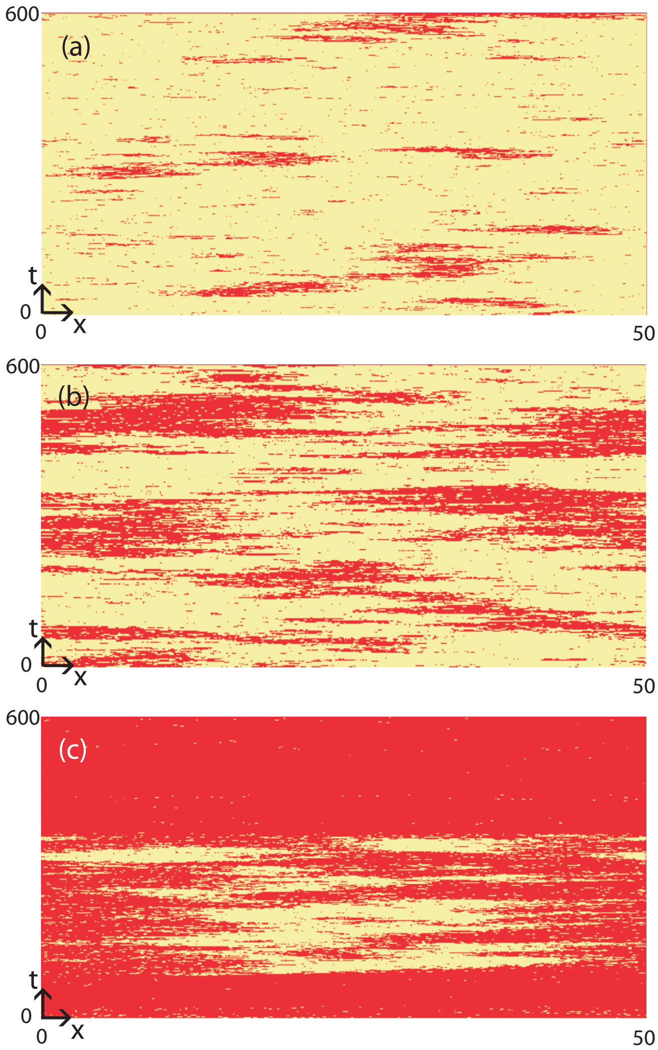

FIG. 3. (Color online) Three typical examples for time series of noisy localized states are presented for $T = 600$ in the asymptotic time regime. Here red (dark gray) and yellow (light gray) indicate values above and below a threshold value of $|A| = 1.5$, respectively. (a) shows the time evolution of an intermittent localized state of finite width emerging and disappearing intermittently as a function of time for $\mu = -1.40$ and $\eta = 0.497$, (b) shows the time evolution of a noisy localized solution for $\mu = -1.40$ and $\eta = 0.514$, of the type shown as a snapshot in Fig. 2(c), while (c) shows the time evolution of a state filling the full width alternating intermittently with a state showing the coexistence between the upper and lower attractor for $\mu = -1.40$ and $\eta = 0.53$.

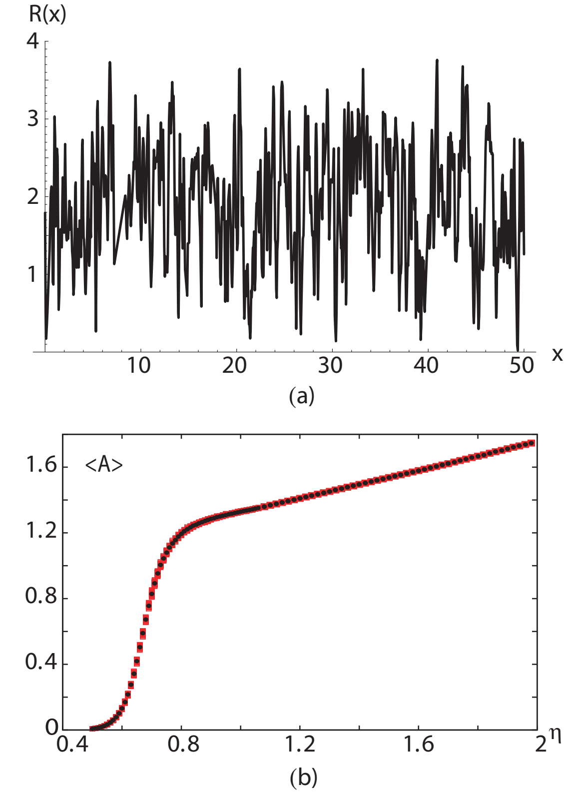

FIG. 4. (Color online) (a) A snapshot of $R(x)$ is shown for $\mu = -1.60$ and $\eta = 1.6$. (b) Plot of $\langle A\rangle$ for $\mu = -1.6$ as a function of η for the range $0.5 < \eta < 2.0$ with 50 realizations for each noise amplitude and $T = 10^4$ for each realization.

the x-t plots immediately that with increasing noise strength the periods of noisy pattern-free solutions are shrinking while the fraction of time with noisy filled-in solutions is increasing. Over a range of noise strengths, as evident also from Fig. 1, a noisy spatially localized solution occurs intermittently as can be seen best in Fig. 3(b). From Figs. 2 and 3 we can also infer typical length and time scales for noise-induced localized states. For example, from inspection of Fig. 2(b) we see that a typical width of a noisy localized state, ΔW, is $\Delta W \sim 10$. This value agrees with typical widths observed for deterministic DSs (compare, for example, Ref. [32]). For the characteristic lifetime, τ, of noisy spatially localized states it is instructive to look at Fig. 3(a): for the associated values of η and μ ($\eta = 0.497$ and $\mu = -1.40$) we estimate a typical lifetime τ: $\tau \sim 30$.

We note that the back and forth dynamics between attractors shown in Fig. 3(c) is not related to stochastic resonance, since our system is not driven by any periodic forcing. In addition, we underscore that our system is neither excitable nor variational.

In Fig. 4(a) we present a snapshot of the modulus in a more strongly subcritical situation ($\mu = -1.60$) for large noise strength $\eta = 1.60$. In Fig. 4(b) we show the plot of $\langle A\rangle$ as a function of noise strength η for the range $0.5 < \eta < 2.0$ with 50 realizations for each noise amplitude for $\mu = -1.60$ and time $T = 10^4$ for each realization. The plot of $\langle A\rangle$ [Fig. 4(b)] shows qualitatively similar behavior as for $\mu = -1.40$. We emphasize, however, that a much higher noise amplitude is necessary to obtain spatial patterns. In addition, we observe that for large noise strength ($\eta > \sim 1.1$) $\langle A\rangle$ grows linearly in η. We note that the dispersion of the data points in the more strongly subcritical region is much smaller than for $\mu = -1.40$ (compare Fig. 1). We trace this result back to the fact that the deterministic domains of attraction for finite amplitude solutions are much narrower and much deeper in magnitude for more strongly subcritical conditions.

Comparing the results given in Figs. 1 and 4 we conclude that for more strongly subcritical values of μ a larger noise strength is needed to induce noisy localized structures. This

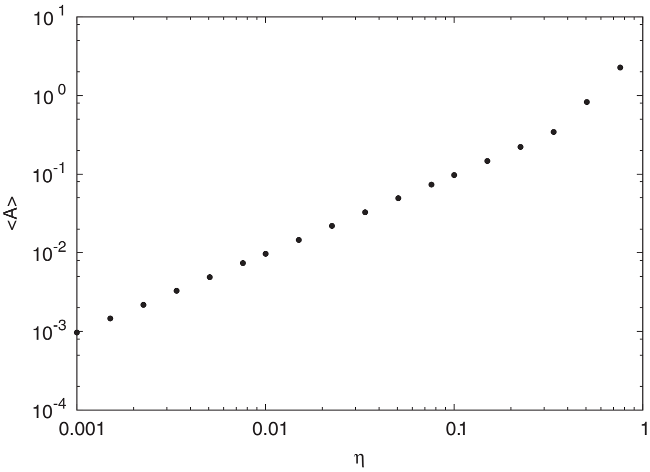

FIG. 5. $\log(\langle A \rangle)$ is plotted as a function of $\log(\eta)$. The power law $\log(\langle A \rangle) \sim \log(\eta)$ is obtained over more than two decades ($0.001 < \eta < 0.1$). $\mu = -1.40$ and without cutoff for $|A|$.

observation is correlated with the fact that the value $|A| \equiv 0$ is more stable and a larger value of η is required to generate a finite pattern amplitude. Finally we note that below the saddle node of the spatially homogeneous deterministic system a finite amplitude branch does not exist. In this case noise only prevails and $\langle A \rangle$ grows linearly with η.

In this Rapid Communication, the intermittent appearance of localization has been quantified by means of $\langle A \rangle$, an order parameter when averaged over a large number of realizations. The η dependence on $\langle A \rangle$, over more than two decades ($0.001 < \eta < 0.1$), is remarkably well fitted by a simple power law: $\log(\langle A \rangle) \sim \log(\eta)$, with an exponent 1.0, as shown in Fig. 5.

In conclusion, we have demonstrated that the influence of large noise can induce the formation of intermittent noisy localized structures. The framework of this study has been the cubic-quintic complex Ginzburg-Landau equation, a nonvariational system with un upper and lower attractor in the deterministic case. The range of parameters in this study has been chosen so that deterministically the system does not show stable pulses. To characterize quantitatively the fraction of the volume filled by a spatial pattern we have introduced the average of the modulus of the pattern amplitude, $|A|$, over space, time, and a large number of realizations: $\langle A \rangle$. Since the variation of this quantity as a function of the noise amplitude is rather insensitive to the cut-off value for $|A|$, to the number of realizations (provided it is sufficiently high), and to the time scales of averaging in the asymptotic regime, $\langle A \rangle$ can be thought of as an order parameter.

Clearly our present study opens the door to several areas of investigation. First of all it will be important to examine to what extent the results presented here can be carried over to other models such as the case of dissipative solitons with energy and matter flows [41] or to dissipative solitons occuring in reaction diffusion systems for which one has shown recently that they can support exploding dissipative solitons [35]. Another key direction to go into is to study the influence of spatial dimensionality of the phenomena described here; this includes dissipative solitons localized in two dimensions [40,42] as well as quasi-one-dimensional DSs [40,43–45].

Natural candidates to investigate the results of our study experimentally are surface reactions like the catalytic oxidation of CO for which one has already a considerable amount of experience [10–15] for the externally controlled superposition of noise on parameters such as partial pressure, flow rate, and temperature. To find experimentally a well controllable bioinspired system to show the type of behavior described here is certainly more of a challenge. One candidate could be the system for which solitonlike structures and their collisions have been observed recently [46].

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